JOURNAL OF APPROXIMATION THEORY **49**, 1–17 (1987)

Further Asymptotic Properties of Best Approximation by Splines

DENNIS D. PENCE

Department of Mathematics, Western Michigan University, Kalamazoo, Michigan 49008, U.S.A.

Communicated by Oved Shisha

Received January 9, 1984; revised April 16, 1985

Let $S_n^{k,r}$ denote the collection of polynomial splines of order k with at most $(n-1)$ free knots, each of multiplicity r. This paper explicitly finds the constants $C_{k,r,p}$ so that

$$\lim_{n\to\infty} n^k \operatorname{dist}_{L_p[0,1]}\{f, S_n^{k,r}\} = C_{k,r,p} \, \|f^{(k)}\|_{L_\sigma[0,1]},$$

where $\sigma = p/(kp+1)$ and f is sufficiently smooth. This completely fills the gap between previously known results for simple knots $(r=1)$ and for piecewise polynomials $(r=k)$. We also consider similar asymptotic properties pertaining to approximation of vector functions by vector splines. We define the latter to be families of vector functions whose components are splines of order k with common knots.

0. INTRODUCTION

The nonlinear problem of finding the *optimal location* for placing $n-1$ knots when trying to best approximate a function f from a family of polynomial spline functions based upon these knots is surprisingly difficult. This paper and several previous ones consider precise asymptotic error bounds as n tends to infinity. The techniques used in the proof of these asymptotic results provide very interesting suggestions for *nonoptimal* knot location schemes and *nonbest* approximation procedures which are much more easily implemented and yet produce the same asymptotic errors as n tends to infinity.

We will consider polynomial splines with multiple knots. Let $S_n^{k,r}$ denote the nonlinear collection of polynomial spline functions defined on $[0,1]$ of order k with at most $n-1$ free knots, each of multiplicity r. Thus $s \in S_n^{k,r}$ if the following hold: (1) there exist knots $\{t_i\}$ with $0 < t_1 \leqslant t_2 < \cdots < t_{n-1} < 1$ (where for simplicity we define $t_0 = 0$ and $t_n = 1$);

0021-9045/87 $3.00

(2) s equals a polynomial of degree $(k-1)$ when restricted to any interval $[t_{i-1}, t_i)$ for $i=1, 2,..., n$; and (3) $s^{(j)}(t_i^-)=s^{(j)}(t_i^+)$ for $j=0, 1,..., k-r-1$ and $i=1, 2,..., n-1$. When $r=1$, $S_n^{k,1}$ denotes polynomial splines with simple knots, and when $r=k$, condition (3) above is empty and $S_n^{k,k}$ denotes the set of piecewise polynomials with no continuity requirement across pieces.

The main result of this paper is to prove that for $1 \leqslant p < \infty$ and for a sufficiently smooth function f, one has

$$\lim_{n \to \infty} n^k \operatorname{dist}_{L_p[0,1]}\{f, S_n^{k,r}\} = C_{k,r,p} \|f^{(k)}\|_{L_\sigma[0,1]},$$

where $\sigma = p/(kp+1)$ and the constants $C_{k,r,p}$ are independent of f and explicitly known. The case when $r=k$ was established by Burchard and Hale [5]. See also the discussion in [12, p. 295] for even earlier work. For $r=1$ and $p=2$, this result can be found in Barrow and Smith [2]. Pence and Smith [10] extended the previous paper to cover when $r=1$ and $1 \leqslant p < \infty$, and they ended with a conjecture on the form of the constants $C_{k,r,p}$ based upon the then-known cases of $r=k$ and $r=1$. We will show that the conjecture there is slightly wrong by proving the correct result.

For completeness, we summarize results about periodic splines with multiple knots which are needed in the proof of our main result. Once we have these preliminary results, the proofs that follow in this paper are not that different from those in [10], so that not every detail will be given. In the last section we give the corresponding results for vector splines, as they are defined there.

1. Preliminary Periodic Spline Results

We bring together some useful results concerning periodic splines. Hopefully this presentation will make more people aware of the usefulness of the shifted Bernoulli functions.

Let $\mathring{S}_m^k(\{z_i\}; \{r_i\})$ denote the linear space of 1-periodic splines of order k. The "$\circ$" reminds us that these functions are really circular. Here we adopt the convention that the knots satisfy

$$0 \leqslant z_1 < z_2 < \cdots < z_m < 1.$$

The multiplicity of knot z_i is specified by the integer r_i where $1 \leqslant r_i \leqslant k$. Taking advantage of the periodicity, we consider these splines to be defined everywhere and we extend the knot set to a biinfinite set $\{z_i\}$, $i=..., -1, 0, 2,....$ Such splines are determined by m polynomial pieces, and the dimension of this space is $d = r_1 + r_2 + \cdots + r_m$. (See [12, Section 8.1] for more details, including zero counting theorems for periodic splines.)

In a certain sense, the analog of the truncated power basis will be provided by the periodic Bernoulli functions. Let

$$B_k(x) = x^k + \binom{k}{1} B_1 x^{k-1} + \binom{k}{2} B_2 x^{k-2} + \cdots + \binom{k}{k-1} B_{k-1} x + B_k$$

denote the kth *Bernoulli polynomial.* The constant term is the kth *Bernoulli number*. These numbers satisfy the recursion relation

$$1 + \binom{k}{1} B_1 + \binom{k}{2} B_2 + \cdots + \binom{k}{k-1} B_{k-1} = 0, \qquad k \geqslant 2,$$

with $B_0 = 1$ for completeness. The key properties of the Bernoulli polynomials are that

$$B_k'(x) = kB_{k-1}(x), \qquad k \geqslant 1; \qquad \int_0^1 B_k(x)\, dx = 0, \qquad k \geqslant 1;$$

and

$$B_k^{(j)}(0) = B_k^{(j)}(1), \qquad j = 0, 1, \ldots, k-2 \quad \text{and} \quad k \geqslant 2.$$

Extending such a Bernoulli polynomial off the interval $[0, 1]$ by periodicity gives the *Bernoulli function* $\mathring{B}_k(x)$. Thus

$$\mathring{B}_k(x) = x^k + \phi_k(x),$$

where $\phi_k(x)$ is a spline of order k with simple knots at the integers [11].

More generally, a function of the form

$$M(x) = [\mathring{B}_k(x - z_1) - \psi(x)]/k!,$$

where $\psi \in \mathring{S}_m^k(\{z_i\}; \{r_i\})$, is called a *periodic monospline*, and we denote the collection of all of these by $M\mathring{S}_m^k(\{z_i\}, \{r_i\})$. Let $M\mathring{S}_m^k(\{r_i\})$ denote the collection with free knots, i.e., the union of all $M\mathring{S}_m^k(\{z_i\}; \{r_i\})$ with $0 \leqslant z_1 < z_2 < \cdots < z_m < 1$. It may happen that a particular monospline has a higher order of continuity across a knot than is required in the class $M\mathring{S}_m^k(\{r_i\})$. For example $\mathring{B}_k(x)/k!$ is a member with $z_1 = 0$ and any $z_2, \ldots, z_m$ satisfying $0 < z_2 < \cdots < z_m < 1$ considered as the knots. For a specific monospline ϕ, we say the *effective multiplicity* of a knot z_i is the integer r, where $0 \leqslant r \leqslant r_i$ and

$$\phi^{(j)}(z_i^-) = \phi^{(j)}(z_i^+), \qquad \text{for} \quad j = 0, 1, \ldots, k - r - 1,$$

but

$$\phi^{(k-r)}(z_i^-) \neq \phi^{(k-r)}(z_i^+), \quad \text{when} \quad r > 0.$$

Combining the results in Section 8.2 and Section 8.4 of [12], we can appeal to a zero counting procedure and a Budan Fourier Theorem for periodic monosplines.

THEOREM A (Bojanov). *For $1 \leqslant p < \infty$, there exists a unique element of $M\mathring{S}_m^k(\{r_i\})$ of minimal norm in $L_p[0, 1]$, ignoring trivial translations of the knot set wholly within $[0, 1)$.*

COROLLARY 1. *In the case where $r_i = r$, for all i, which we will denote by $M\mathring{S}_m^{k,r}$, the knots of the minimal periodic monospline of Theorem A are equally spaced and have effective multiplicity equal to r if r is odd, and equal to $r-1$ if r is even.*

The case where $r = 1$ of the corollary can be found within [13]. The theorem and the corollary are explicitly stated in [4] where topological degree theory is used to establish uniqueness. Both Zensykbaev and Bojanov explore the connections with optimal quadrature rules. We reprove the corollary and investigate the precise form of the minimal periodic monospline to obtain the bounds needed later.

Proof of Corollary 1. We can represent an arbitrary periodic monospline $M(x) \in MS_m^{k,r}$ by

$$k!\, M(x) = G(x) = \sum_{i=1}^{m} \sum_{j=0}^{r-1} a_{ij} \mathring{B}_{k-j}(x - z_i) + c,$$

where we require that

$$\sum_{i=1}^{m} a_{i,0} = 1.$$

The necessary conditions for a minimum for $\|G(x)\|_p^p$ to be realized by $G^*(x)$ by varying the coefficients and knots subject to the above equality constraint are the following (using Lagrange multipliers and then simplifying):

(1) $\int_0^1 |G^*(x)|^{p-1} \operatorname{sgn} G^*(x)\, dx = 0;$

(2) $\int_0^1 |G^*(x)|^{p-1} \operatorname{sgn} G^*(x)\, \mathring{B}_k(x - z_i)\, dx = \lambda, \qquad i = 1,..., m;$

(3) $\int_0^1 |G^*(x)|^{p-1} \operatorname{sgn} G^*(x)\, \mathring{B}_{k-j}(x - z_i)\, dx = 0,$
$i = 1,..., m$ and $j = 1,..., r-1;$

(4) $a_{i,(r-1)} \int_0^1 |G^*(x)|^{p-1} \operatorname{sgn} G^*(x)\, \mathring{B}_{k-r}(x - z_i)\, dx = 0,$
$i = 1,..., m$ when $r < k$

or $a_{i,(k-1)} |G^*(z_i)|^{p-1} \operatorname{sgn} G^*(z_i) = 0$, $i = 1,..., m$ when $r = k$.

Note. The partial differentiation with respect to z_i giving (4) when $r=k$ above is accomplished in a manner similar to the differentiation of $g^{(k-1)}$ immediately below.

Following the arguments of [13], let

$$g(t)=\frac{\lambda}{p}-\int_0^1 |G^*(x)|^{p-1}\operatorname{sgn} G^*(x)\, \mathring{B}_k(x-t)\,dx.$$

Then

$$g^{(j)}(t)=\frac{(-1)^j\,k!}{(k-j)!}\int_0^1 |G^*(x)|^{p-1}\operatorname{sgn} G^*(x)\,\mathring{B}_{k-j}(x-t)\,dx,$$

$$j=1,2,\ldots,k-2;$$

$$g^{(k-1)}(t)=(-1)^{k-1}\,k!\int_t^{1+t} |G^*(x)|^{p-1}\operatorname{sgn} G^*(x)\,B_1(x-t)\,dx;$$

$$g^{(k)}(t)=(-1)^{k-1}\,k!\,|G^*(t)|^{p-1}\operatorname{sgn} G^*(t).$$

Thus $g^{(k)}$ and G^* have the same zeros (not counting multiplicity). Conditions (2) and (3) imply that

$$g^{(j)}(z_i)=0, \qquad \text{for} \quad i=1,\ldots,m \quad \text{and} \quad j=0,1,\ldots,r-1.$$

Condition (4) implies that either $a_{i,(r-1)}=0$ or $g^{(r)}(z_i)=0$, where $i=1,\ldots,m$. Note that when $a_{i,(r-1)}=0$, the effective multiplicity of the knot z_i drops down to at most $r-1$. Let $\mathbf{I}$ denote the set of indices i where $a_{i,(r-1)}\neq 0$, and let I denote the cardinality of $\mathbf{I}$. Applying Rolle's Theorem, we can conclude that $g^{(k)}$ (and hence G^*) has at least $rm+I$ simple zeros. However, the monospline $G^*(x)/k!$ can have at most rm zeros when r is even and at most $rm+2I-m$ zeros when r is odd. This follows from the strong version of the Budan–Fourier Theorem for monosplines [12, Theorem 8.43, p. 332], considering that $G^*(x)/k!$ is *periodic* and is a member of the set of monosplines with multiplicity r at each knot z_i where $i\in\mathbf{I}$ and multiplicity $(r-1)$ at each knot z_i where $i\notin\mathbf{I}$. Thus we must have that $I=0$ when r is even and $I=m$ when r is odd. That means that the effective multiplicity of each knot in $G^*(x)$ must be odd, regardless of the parity of r. It sufficies to consider further only the case when r is odd (since this same monospline G^* must be optimal in the case of the even multiplicity one greater than r).

Theorem A guarantees a unique minimal $G^*(\cdot)$ with, say $z_1=0$. However, each $G^*(\cdot-z_i)$ will also be a candidate having the same norm. We conclude that the knots must be equally spaced and that the coefficients of like orders of shifted Bernoulli functions in G^* must be equal, i.e.,

$$z_{i+1} = i/m, \qquad i = 0, 1, \ldots, m-1$$

and

$$a_{ij} = \alpha_j, \qquad j = 0, 1, \ldots, r-1 \quad \text{and} \quad i = 1, \ldots, m.$$

In particular, $\alpha_0 = 1/m$ since $\sum a_{i,0} = 1$. Use the identity (see [1, 8])

$$B_v(mx) = m^{v-1} \sum_{i=0}^{m-1} B_v(x + i/m).$$

Then for $0 \leqslant x < 1/m$,

$$B_v(mx) = m^{v-1} \left[B_v(x) + \sum_{i=1}^{m-1} \mathring{B}_v\left(x - \frac{m-i}{m} \right) \right] = m^{v-1} \left[\sum_{i=0}^{m-1} \mathring{B}_v(x - i/m) \right]$$

and

$$G^*(x) = \frac{1}{m} \frac{B_k(mx)}{m^{k-1}} + \sum_{j=1}^{r-1} \frac{\alpha_j}{m^{k-j-1}} B_{k-j}(mx) + c.$$

Since shifting by $1/m$ leaves G^* unchanged, we have

$$\begin{aligned} \|G^*(x)\|_p^p &= \int_0^1 |G^*(x)|^p \, dx = \sum_{l=0}^{m-1} \int_{l/m}^{(l+1)/m} |G^*(x)|^p \, dx = m \int_0^{1/m} |G^*(x)|^p \, dx \\ &= m \int_0^{1/m} \left| \frac{B_k(mx)}{m^k} + \sum_{j=1}^{r-1} \frac{\alpha_j}{m^{k-j-1}} B_{k-j}(mx) + c \right|^p dx \\ &= m^{-pk} \int_0^1 \left| B_k(x) + \sum_{j=1}^{r-1} \tilde{\alpha}_j B_{k-j}(x) + \tilde{c} \right|^p dx, \end{aligned}$$

where $\tilde{\alpha}_j = m^{j+1} \alpha_j$ and $\tilde{c} = m^k c$.

Further, these coefficients must be chosen to make the norm of G^* as small as possible. We define

$$C_{k,r,p} = \frac{1}{k!} \min_{\alpha, c} \left\| B_k(\cdot) + \sum_{j=1}^{r-1} \alpha_j B_{k-j}(\cdot) + c \right\|_p.$$

If r is odd, $C_{k,r+1,p} = C_{k,r,p}$ since, as noted above, the same monospline $G^*(x)$ will satisfy the conditions of optimality for the case r and the case $r+1$.

Then the minimal L_p-norm monospline M^* from $M\mathring{S}_m^{k,r}$ satisfies

$$\|M^*(\cdot)\|_p = m^{-k} C_{k,r,p}, \qquad 1 < p < \infty, \ 1 \leqslant r < k.$$

When $p = 2$, these constants can be easily calculated using that

$$\int_0^1 B_v(x)\, B_j(x)\, dx = (-1)^{(j-1)} \frac{v!\, j!}{(v+j)!} B_{v+j}, \qquad j \leqslant v:$$

$$\begin{aligned}
&C_{1,1,2} = \sqrt{1/12} = 0.28868 \\
&C_{2,1,2} = \sqrt{1/180} = 0.07454 \\
&C_{3,1,2} = \sqrt{1/840} = 0.03450 \qquad && C_{3,3,2} = \sqrt{1/2800} = 0.01890 \\
&C_{4,1,2} = \sqrt{1/2100} = 0.02182 \qquad && C_{4,3,2} = \sqrt{1/44100} = 0.00476 \\
&C_{5,1,2} = \sqrt{5/16632} = 0.01734 \qquad && C_{5,3,2} = \sqrt{1/332640} = 0.00173,
\end{aligned}$$

which is the table of values for $C_{k,r,2}$.

2. Asymptotic Properties for Spline Approximation

We can now completely settle the conjecture given at the very end of [10] where it is conjectured that

$$C_{k,r,p} = \frac{1}{k!} \min_{\bar{a}} \| B_k(x) - \{a_0 + a_1 + \cdots + a_{r-1} x^{r-1}\} \|_p.$$

Given a sequence of knots $\{t_i\} = \{0 = t_0 < t_1 < \cdots < t_n = 1\}$ for the interval $[0, 1]$, let $S_n^{k,r}(\{t_i\})$ denote the space of polynomial splines of order k with knots $\{t_i\}$, each of multiplicity r, where $1 \leqslant r \leqslant k$. A special method for obtaining knots is to use a knot quantile function t, where the t maps $[0, 1]$ onto itself with $t' \geqslant 0$. Let $S_n^{k,r}(t) = S_n^{k,r}(\{t(i/n)\},\ i = 0, \ldots, n)$. Recall that $S_n^{k,r}$ denotes the collection of splines of order k with at most $(n-1)$ free knots, each of multiplicity r, and adding a "$\circ$" above indicates adding the adjective *periodic* (on $[0, 1]$).

Theorem 1. *Let $f \in C^k[0, 1]$, the bijection $t \in C^1[0, 1]$ satisfy $0 < \delta < t'$ and $1 \leqslant p < \infty$. Then*

$$\lim_{n \to \infty} n^k \operatorname{dist}_{L_p[0,1]}(f, S_n^{k,r}(t)) = C_{k,r,p} (J_{k,p}(f, t))^{1/p},$$

where

$$J_{k,p}(f, t) = \int_0^1 |f^{(k)}(t(x))|^p (t'(x))^{kp+1}\, dx,$$

and the constant $C_{k,r,p}$ is defined in Section 1.

The special case of Theorem 1 when $r=1$ and $p=2$ was shown by Barrow and Smith [2]. The case when $r=1$ and $1 \leqslant p < \infty$ can be found in [10], where the results were further extended to cover when $t' \geqslant 0$ and the set of points where either $f^{(k)}$ or t' fails to exist has content zero. The same extensions can be made for the results of this paper, but for simplicity this will not be discussed. The methods used in [10] carry over with little change, and we note only the major differences in the outlines of the proof.

Outline of the Proof of Theorem 1

Let $P_n^{k,r,p}(t)$ denote the $L_p[0,1]$ metric projection onto $S_n^{k,r}(t)$ and $P_n^{k,r,p}(\{z_i\})$ denote the projection onto $S_n^{k,r}(\{z_i\})$. Suppose I is a closed subinterval of $[0,1]$ and $\tau\colon [0,1] \to I$ denotes the linear change of variable, i.e., τ' is constantly $|I|$, the length of the interval I. Let $\bar{f}$ denote the kth degree Taylor expansion of f about some point $a \in I$ and R denote the remainder function. Suppose that only the $l-1$ points $t_{l_0+1},\dots, t_{l_0+l-1}$ from $\{t(i/n)\}$ lie in the interior of I and that $\tau(z_{i'}^*) = t_{l_0+i'} = 1,\dots, l-1$. Further let $z_0^* = 0$ and $z_l^* = 1$. Then

$$\begin{aligned} l^k \, \|f - P_n^{k,r,p}(t)\, f\|_{p,I} &= l^k \, |I|^{1/p} \, \|f \circ \tau - (P_n^{k,r,p}(t)\, f) \circ \tau\|_{p,[0,1]} \\ &\geqslant l^k \, |I|^{1/p} \{ \|\bar{f} \circ \tau - (P_n^{k,r,p}(t)\, f) \circ \tau\|_{p,[0,1]} - \|(f - \bar{f}) \circ \tau\|_{p,[0,1]} \} \\ &\geqslant l^k \, |I|^{1/p} \{ \bar{f} \circ \tau - (P_n^{k,r,p}(\{z_i^*\}))(\bar{f} \circ \tau)\|_{p,[0,1]} - \|R \circ \tau\|_{p,[0,1]} \}. \end{aligned}$$

Let $\Omega(f^{(k)}, a, I) = \max\{|f^{(k)}(\xi) - f^{(k)}(a)|, \xi \in I\}$, a quantity often used in remainder estimates. Then we can bound the remainder R and conclude that

$$l^k \, |I|^{1/p} \, \|R \circ \tau\|_{p,[0,1]} = \|R\|_{p,I} < \frac{2 l^k \, |I|^{(kp+1)/p}}{k!(kp+1)} \, \Omega(f^{(k)}, a, I).$$

We can interpret any spline on $[0,1]$ as a periodic spline where 0 is added as a k-tuple knot. Further we recognize that $B_k(\cdot)$ is a monic polynomial. Thus we can use the results of Section 1 in a manner similar to [10, p. 410]:

$$\begin{aligned} & l^k \, |I|^{1/p} \, \|\bar{f} \circ \tau - (P_n^{k,r,p}(\{z_i^*\}))(\bar{f} \circ \tau)\|_{p,[0,1]} \\ & \geqslant l^k \, |I|^{1/p} \inf_{\psi \in S_{l+2k}^{k,r}} \|f^{(k)}(a)\, |I|^k \cdot B_k(\cdot) - \psi\|_{p,[0,1]} \\ & \geqslant \left(\frac{l}{l+2k}\right)^k |I|^{k+(1/p)} \, |f^{(k)}(a)| \, C_{k,r,p}. \end{aligned}$$

Let l be an arbitrary positive integer. For each $n > l$, let m_n be the greatest integer less than or equal to (n/l), $I_j^n = [(j-1)\,l/n, jl/n]$ and $J_j^n = t(I_j^n)$, for $j = i,\dots, m_n$. Then following the argument in [10, p. 411],

$$\begin{aligned}
&n^{kp}(\| f - P_n^{k,r,p}(t) f \|_{p,[0,1]})^p \\
&\geqslant \left(\frac{n}{l}\right)^{kp} \sum_{j=1}^{m_n} l^{kp}\Big(\| f - P_n^{k,r,p}(t) f \|_{p,J_j^n})^p \\
&\geqslant \left(\frac{n}{l}\right)^{kp} \sum_{j=1}^{m_n} \left| \left(\frac{l}{l+2k}\right)^k C_{k,r,p} | f^{(k)}(\xi_j^n)| - K_1 l^k \Omega(f^{(k)}, \xi_j^n, J_j^n) \right|^p |J_j^n|^{kp+1} \\
&\geqslant \left(\frac{n}{l}\right)^{kp} \sum_{j=1}^{m_n} \left[\left(\frac{l}{l+2k}\right)^{kp} C_{k,r,p}^p \, | f^{(k)}(\xi_j^n)|^p - K_2 l^k \Omega(f^{(k)}, \xi_j^n, J_j^n) \right] |J_j^n|^{kp+1}.
\end{aligned}$$

There exist points $\eta_j^n \in I_j^n$ such that

$$|J_j^n| = t'(\eta_j^n)\left(\frac{l}{n}\right).$$

Since the above estimates are valid for any choice of $\xi_j^n \in J_j^n$, we set $\xi_j^n = t(\eta_j^n), j = 1,\dots, m_n$.

Then

$$\begin{aligned}
&n^{kp}(\| f - P_n^{k,r,p}(t) f \|_{p,[0,1]})^p \\
&\quad\geqslant \sum_{j=1}^{m_n} \left(\frac{l}{l+2k}\right)^{kp} C_{k,r,p}^p \, | f^{(k)}(t(\eta_j^n))|^p (t'(\eta_j^n))^{kp+1} \left(\frac{l}{n}\right) \\
&\quad\quad - \sum_{j=1}^{m_n} K_2 l^k \Omega(f^{(k)}, \xi_j^n, J_j^n)(t'(\eta_j^n))^{kp+1} \left(\frac{l}{n}\right).
\end{aligned}$$

The first summation lacks only a term for the subinterval $[m_n(l/n), 1]$ in order to be a Riemann sum for the interval $[0, 1]$. Thus as $n \to \infty$, it tends to

$$\left(\frac{l}{l+2k}\right)^{kp} C_{k,r,p}^p \int_0^1 | f^{(k)}(t(x))|^p (t'(x))^{kp+1}\, dx.$$

The second summation is bounded above by

$$K_2 l^{k+1} \omega(f^{(k)}, \{\max t'\}(l/n))\{\max t'\}^{kp+1}(m_n/n).$$

This goes to zero as $n \to \infty$ because $f^{(k)}$ is continuous and its modulus of continuity, $\omega(f^{(k)}, h) = \sup\{| f^{(k)}(x) - f^{(k)}(y)|\colon |x-y| \leqslant h\}$, tends to zero as $h \to 0$.

Therefore

$$\varliminf_{n\to\infty} n^{kp}(\|f-P_n^{k,r,p}(t)f\|_{p,[0,1]})^p \geqslant \left(\frac{l}{l+2k}\right)^{kp} C_{k,r,p}^p J_{k,p}(f,t).$$

Taking the limit as $l\to\infty$, we obtain a lower bound for the desired quantity.

An upper bound for

$$\varlimsup_{n\to\infty} n^{kp}\,\|f-P_n^{k,r,p}(t)f\|_{p,[0,1]}^p$$

is obtained by replacing the best approximation $P_n^{k,r,p}(t)f$ by a locally defined approximation

$$Q_n f = \sum_v \lambda_{v,n}(f)\,N_{v,n},$$

where $\{z_v\}$ is the knot set $\{t(i/n)\}$ with each interior knot repeated r times and the end points 0 and 1 repeated as needed, and $N_{v,n}$ denotes the k-order normalized B-spline based upon $\{z_n\}$ with supp $N_{v,n}=[z_v, z_{v+k}]$. The B-spline coefficients are determined as follows. Let $\mathbf{J}$ be the set of indices satisfying (1) $0\in\mathbf{J}$; (2) if $j\in\mathbf{J}$, then $\tilde{j}=\min\{m\colon m\geqslant j+k, z_m=z_{m+1}\}\in\mathbf{J}$. For each $j\in\mathbf{J}$, we require that

$$\bar{f}_j(\tau)-(Q_n f)(\tau)=f_j^k\,B_{k,r,p}\left(\frac{\tau-z_j}{h_j}\right)(h_j)^k,$$

for

$$z_j\leqslant\tau\leqslant z_{j+1}=z_j+h_j,$$

where

$$\bar{f}_j(\tau)=\sum_{s=0}^{k} f_j^s(\tau-z_j)^s=\sum_{s=0}^{k}\left[\frac{f^{(s)}(z_j)}{s!}\right](\tau-z_j)^s,$$

and $B_{k,r,p}(x)$ is the monic polynomial providing the minimum in the definition of

$$C_{k,r,p}=\frac{1}{k!}\min_{\alpha,c}\|B_k(\ )+\alpha_1 B_{k-1}(\ )+\cdots+\alpha_{r-1}B_{k-r-1}(\ )+c\|_{p,[0,1]}.$$

Any coefficient function $\lambda_{v,n}$ not specified above is set to zero. The justification that this provides the desired upper bound is similar to the simple knot arguments in [10] and was briefly presented in [9]. Thus Theorem 1 is established.

We can now apply a calculus of variation argument to the bound in Theorem 1 to obtain an optimal knot quantile function. This is easily done by changing the variational problem. Let $u(\tau)=(t^{-1})'(\tau)$. Then

$$J_{k,p}(f, t)=\int_0^1 |f^{(k)}(\tau)|^p (u(\tau))^{-kp}\, d\tau.$$

The optimal function u^* minimizing this integral is

$$u^*(\tau)=|f^{(k)}(\tau)|^\sigma \Big/ \int_0^1 |f^{(k)}(\xi)|^\sigma\, d\xi, \qquad \sigma=p/(kp+1).$$

Then the associated optimal quantile function t^* (in the case where we assume $|f^{(k)}|$ is never zero) makes the integrand in

$$J_{k,p}(f, t^*)=\int_0^1 |f^{(k)}(t^*(x))|^p (t^{*\prime}(x))^{kp+1}\, dx$$

a constant function. Thus the terms in the following summations are all equal.

$$\begin{aligned} J_{k,p}(f, t^*) &= \sum_{i=0}^{n-1} \int_{i/n}^{(i+1)/n} |f^{(k)}(t^*(x))|^p (t^{*\prime}(x))^{kp+1}\, dx \\ &= \|f^{(k)}\|_{\sigma,[0,1]}^{p-\sigma} \sum_{i=0}^{n-1} \int_{t^*(i/n)}^{t^*((i+1)/n)} |f^{(k)}(\tau)|^\sigma\, d\tau. \end{aligned}$$

This justifies schemes to locate knots $\{t_i\}$ so as to equalize

$$\int_{t_i}^{t_{i+1}} |f^{(k)}(\tau)|^\sigma\, d\tau \approx K(n)\, \|f-P_n^{k,r,p}(\{t_i\})\|_{p,[t_i,t_{i+1}]}^p,$$

i.e., to "balance" the errors on knot intervals.

The above argument provides part of the proof of the following free knot result. Since the complete proof is very similar to that given in [2, pp. 229–302] and [10, pp. 417–420] and since we will be considering a generalized version in the next section, we will omit this proof.

THEOREM 2. *For* $f \in C^k[0,1]$,

$$\lim_{n\to\infty} n^k \operatorname{dist}_{L_p[0,1]}(f, S_n^{k,r}) = C_{k,r,p}\, \|f^{(k)}\|_{\sigma,[0,1]} \qquad \text{where } \sigma=(p/(kp+1)).$$

3. Vector Approximations

It is easy to carry over the previous work to the setting of approximating vector functions by vector splines. Let $VS_n^{k,r}(\{t_v\})$ denote the space of vector splines of the form $\mathbf{s} = (s_1, ..., s_m)$, where each $s_i \in S_n^{k,v}(\{t_v\})$, i.e., the component functions are splines which share a common knot set. The vector generalization of the various spline collections will be denoted in a similar fashion. Of the many possible norms on vector functions $\mathbf{f} = (f_1, ..., f_m)$, we choose

$$\|\mathbf{f}\|_p = \left[\sum_{i=1}^{m} w_i \|f_i\|_p^p\right]^{1/p} = \left[\int_0^1 \sum_{i=1}^{m} w_i |f_i(x)|^p \, dx\right]^{1/p},$$

where the weights $\{w_i\}$ are strictly positive real numbers. The space of vector functions for which this norm is finite will be denoted by $VL_p[0, 1]$, with the usual convention of identifying equivalence classes to obtain a true norm. Note that

$$\|\mathbf{f}\|_p = \left\{\int_0^1 \|\mathbf{f}(x)\|_{\#,p}^p \, dx\right\}^{1/p},$$

where $\|(y_1, ..., y_m)\|_{\#,p} = \{\sum_{i=1}^m w_i |y_i|^p\}^{1/p}$, a weighted norm in $l_p(m)$.

To give a practical example where this type of approximation might be desirable, the attitude, or rotational orientation, of a satellite can be represented in a variety of ways as a vector function where time is the independent variable. For instance a 3–1–2 Euler angle sequence $(\psi(x), \phi(x), \theta(x))$ would describe the succession of rotation angles of appropriate coordinate axes to go from some reference frame to the orientation of the satellite at time x. While each component function could be approximated independently by splines, the storage requirements for saving the approximations can be reduced by one-third if a common knot set is used.

THEOREM 3. *Let* $\mathbf{f} = (f_1, ..., f_m)$, *where* $f_i \in C^k[0, 1]$, *for* $i = 1, ..., m$; *and let* $t \in C'[0, 1]$ *be a bijection satisfying* $0 < \delta < t'$. *Then*

$$\lim_{n \to \infty} n^k \operatorname{dist}_{VL_p[0,1]}(\mathbf{f}, VS_n^{k,r}(t)) = C_{k,r,p}[J_{k,p}(\mathbf{f}, t)]^{1/p},$$

where

$$J_{k,p} \|(\mathbf{f}, t) = \int_0^1 \sum_{i=1}^m (w_i | f_i^{(k)}(t(x))|^p)(t'(x))^{kp+1}\, dx.$$

Proof

$$n^{kp} \| \mathbf{f} - \mathbf{s} \|_p^p = n^{kp} \int_0^1 \left(\sum_{i=1}^m w_i \, | f_i(x) - s_i(x)|^p \right) dx$$

$$= \sum_{i=1}^m w_i \, n^{kp} \int_0^1 | f_i(x) - s_i(x)|^p \, dx$$

Applying Theorem 1 to each component gives the desired result.

THEOREM 4. *Let* $\mathbf{f} = (f_1, ..., f_m)$, *where* $f_i \in C^k[0, 1]$, *for* $i = 1, ..., m$. *Then*

$$\lim_{n \to \infty} n^k \operatorname{dist}_{VL_p[0,1]}(\mathbf{f}, VS_n^{k,r}) = C_{k,r,p}\, W(f_1^{(k)}, ..., f_m^{(k)}),$$

where

$$W(f_1^{(k)}, ..., f_m^{(k)}) = \left[\int_0^1 \left(\sum_{i=1}^m w_i \, | f_i^{(k)}(\tau)|^p \right)^{1/(kp+1)} d\tau \right]^{(kp+1)/p}.$$

Proof: *Part 1* (*upper bound*). It is advantageous to rewrite $J_{k,p}(f, t)$ as

$$\mathbf{J}(u) = \int_0^1 \left\{ \sum_{i=1}^m w_i \, | f_i^{(k)}(\tau)|^p \right\} u(\tau)^{-kp} \, d\tau,$$

where $u(\tau) = (t^{-1})'(\tau)$. Then

$$\int_0^\tau u(z)\, dz = \int_0^\tau (t^{-1})'(z)\, dz = t^{-1}(\tau) - t^{-1}(0) = x,$$

when $t(x) = \tau$. Thus the restrictions on the function u must be that

$$u(\tau) \geqslant 0, \qquad \text{for } 0 \leqslant \tau \leqslant 1 \quad \text{and} \quad \int_0^1 u(\tau d)\, \tau = 1.$$

A standard variational argument yields that the only critical function is

$$u^*(\tau) = \left(\sum_{i=1}^m w_i \, | f^{(k)}(\tau)| \, p \right)^{1/(kp+1)} \bigg/ \int_0^1 \left(\sum_{i=1}^m w_i | f_i^{(k)}(z)|^p \right)^{1/(kp+1)} dz.$$

Checking the second variation verifies that this indeed yields a minimum.

Letting t^* denote the knot quantile function associated with u^*,

$$\begin{aligned}\mathbf{J}(u^*) &= J_{k,p}(f, t^*)\\ &= \sum_j \int_{t^*(j/n)}^{t^*((j+1)/n)} \left(\sum_{i=1}^m w_i\,|f_i^{(k)}(\tau)|^p\right)^{1/(kp+1)} d\tau\\ &\quad\times\left[\int_0^1 \left(\sum_{i=1}^m w_i\,|f_i^{(k)}(z)|^p\right)^{1/(kp+1)} ds\right]^{kp}\end{aligned}$$

Again the optimal knot quantile function t^* makes the original integrand in $J_{k,p}(\mathbf{f}, t^*)$ constant, and this can be interpreted as balancing the errors over knot intervals, i.e., the terms for $j=0,..., n-1$ above are equal.

This establishes the desired upper bound.

$$\begin{aligned}\overline{\lim}_{n\to\infty} n^k \operatorname{dist}_{VL_p[0,1]}(\mathbf{f}, VS_n^{k,r}) &\leqslant \inf_t \lim_{n\to\infty} n^k \operatorname{dist}_{VL_p[0,1]}(\mathbf{f}, VS_m^{k,r}(t))\\ &= C_{k,r,p}(J_{k,p}(\mathbf{f}, t^*))^{1/p} = C_{k,r,p}\, W(f_1^{(k)},..., f_m^{(k)}).\end{aligned}$$

Part 2 (*lower bound*). We establish the claim that

$$\underline{\lim}_{n\to\infty} n^k \operatorname{dist}_{VL_p[0,1]}(\mathbf{f}, VS_n^{k,r}) \geqslant C_{k,r,p}\, W(f_1^{(k)},..., f_m^{(k)}). \tag{3.1}$$

The proof closely follows that in [2, Theorem 2, pp. 300–302] and in [10, Theorem 5.2, pp. 418–210].

Case (i). Let $\mathbf{f}(x) = (\beta_1 x^k, \beta_2 x^k,..., \beta_m x^k)/k!$ Then

$$\begin{aligned}&[n^k \operatorname{dist}_{VL_p[0,1)}(\mathbf{f}, VS_n^{k,r})]^p\\ &\quad= [n^k \operatorname{dist}_{VL_p[0,1]}((\beta_1 B_k(\cdot),..., \beta_m B_k(\cdot))/k!, VS_n^{k,r})]^p\\ &\quad\geqslant [n^k \operatorname{dist}_{VL_p[0,1]}((\beta_1 B_k(\cdot),..., \beta_m B_k(\cdot)/k!, VS_{n+2k}^{k,r})]^p\\ &\quad= n^{kp}\int_0^1 \left(\sum_{i=1}^m w_i\,|\beta_i|^p\,|B_{k,r,p}(x)|^p/(k!)^p\right) dx\\ &\quad= \frac{n^{kp} C_{k,r,p}^p}{(n+2k)^{kp}} \sum_{i=1}^m w_i\,|\beta_i|^p.\end{aligned}$$

Notice

$$\begin{aligned}\sum_{i=1}^m w_i\,|\beta_i|^p &= \sum_{i=1}^m w_i\,|f_i^{(k)}(\tau)|^p\\ &= \left[\int_0^1 \left(\sum_{i=1}^m w_i\,|f_i^{(k)}(\tau)|^p\right)^{1/(kp+1)} d\tau\right]^{kp+1}\\ &= [W(\beta_1,..., \beta_m)]^p.\end{aligned}$$

Then

$$\varliminf_{n \to \infty} n^k \operatorname{dist}_{VL_p[0,1]}(\mathbf{f}, VS_m^{kr}) \geqslant \lim_{n \to \infty} \left(\frac{n}{n+2k}\right)^k C_{k,r,p}\, W(\beta_1, ..., \beta_m)$$

$$= C_{k,r,p}\, W(\beta_1, ..., \beta_m).$$

Therefore (3.1) is valid in this case.

Case (ii). Let $\mathbf{f} = (f_1, ..., f_m)$, where $f_i \in C^k[0, 1]$ and $\sum_{i=1}^m w_1 \, |f_i^{(k)}|^p \geqslant \delta > 0$. Suppose the lower bound (3.1) is false. Then for some infinite subset of positive integers Z_1, we have

$$n^k \operatorname{dist}_{VL_p[0,1]}(\mathbf{f}, VS_n^{kr}) =: n^k \, \|\mathbf{f} - \mathbf{s}_n\| < dW(f_1^{(k)}, ..., f_m^{(k)}), \tag{3.2}$$

where $0 < d < C_{k,r,p}$ and $n \in Z_1$. For each $L = 1, 2, ...,$ and for n sufficiently large in Z_1, subdivide the interval $[0, 1]$ into finitely many closed subintervals $\{I_v := [\alpha_v, \alpha_{v+1}]\}$ whose endpoints coincide with the knots of the optimal $\mathbf{s}_n$, where $L \leqslant L_v \leqslant L + k + 1$. Thus $\sum L_v =: \bar{n} \leqslant n$. Let $\sigma = p/(kp+1)$. The inequality (3.2) implies that

$$n^{k\sigma} \left(\int_0^1 \sum_{i=1}^m w_i f_i - s_{n,i}|^p\right)^{1/(kp+1)}$$

$$< d^\sigma \int_0^1 \left(\sum_{i=1}^m w_i \, |f_i^{(k)}(\tau)|^p\right)^{1/(kp+1)} d\tau. \tag{3.3}$$

Suppose that for *every* v, we have

$$d^\sigma \int_{I_v} \left(\sum_{i=1}^m w_i \, |f_i^{(k)}(\tau)|^p\right)^{1/(kp+1)} d\tau$$

$$\leqslant L_v^{k\sigma} \left(\int_{I_v} \sum_{i=1}^m w_i \, |f_i - s_{n,i}|^p\right)^{1/(kp+1)} \tag{3.4}$$

If so, then summing both sides over v and applying Holder's inequality for finite sequences yields

$$d^\sigma \int_0^1 \left(\sum_{i=1} w_i \, |f_i^{(k)}(\tau)|^p\right)^{1/(kp+1)} d\tau$$

$$\leqslant \sum_v L_v^{k\sigma} \left(\int_{I_v} \sum_{i=1}^m w_i \, |f_i - s_{n,i}|^p\right)^{1/(kp+1)}$$

$$\leqslant \left\{\sum_v L_v\right\}^{k\sigma} \left\{\sum_v \int_{I_v} \sum_{i=1}^m w_i \, |f_i - s_{n,i}|^p\right\}^{1/(kp+1)}$$

$$\leqslant \bar{n}^{k\sigma} \left(\int_0^1 \sum_{i=1}^m w_i \, |f_i - s_{n,i}|^p\right)^{1/(kp+1)}. \tag{3.5}$$

However, (3.5) would contradict (3.3). Thus for every positive integer $n \in Z_1$, there exists an index $v = v_n$ where (3.4) *fails* to be valid.

Next we pass to an infinite subset $Z_2 \subset Z_1$ where we have every $L_{v_n} = \bar{L}$, some constant, when $n \in Z_2$. Further, let a be an accumulation point of the set of left endpoints of I_{v_n}, $n \in Z_3$. Again passing to an infinite subset $Z_3 \subset Z_2$, we can have a as the limit point of α_{v_n}, $n \in Z_3$. Since $\sum w_i |f_i^{(k)}|^p \geqslant \delta > 0$, we can argue that the length of I_{v_n} must go to zero as n from Z_3 goes to infinity. Now for $n \in Z_3$,

$$\bar{L}_{v_n}^k \left(\int_{I_{v_n}} \sum w_i |f_i - s_{n,i}|^p \right)^{1/p} < d \left(\int_{I_{v_n}} \left(\sum w_i |f^{(i)}|^p \right)^{1/(kp+1)} \right)^{1/\sigma}$$

changes variables from I_{v_n} to $[0, 1]$ on both sides:

$$\bar{L}^k \left(\int_0^1 \sum w_i |f_i(|I_{v_n}| z + \alpha_{v_n}) - \tilde{s}_i(z)|^{1/p} \right)^p$$
$$< d \left(\int_0^1 \left(\sum w_i \left| f_i^{(k)}(|I_{v_n}| z + \alpha_{v_n}) \right|^p \right)^{1/(kp+1)} \right)^{1/\sigma}.$$

Take the limit as n from Z_3 tends to infinity:

$$\bar{L}^k \left(\int_0^1 \sum w_i |\beta_i z^k/k! - \tilde{\tilde{s}}_i(z)^p \, dz \right)^{1/p} < dW(\beta_1, ..., \beta_m),$$

where $\beta_i = |f_i^{(k)}(a)|$, $i = 1, ..., m$. However, this contradicts case (i) if such is possible for any $L < \bar{L}$. Thus we conclude that the lower bound (3.1) is valid in this second case.

Case (iii). Let $\mathbf{f} = (f_1, ..., f_m)$, where $f_i \in C^k[0, 1]$, for $i = 1, ..., m$, and let $A = \{x \in [0, 1]: \sum_{i=1}^m w_i f_i^{(k)}(x) = 0\}$. Suppose (3.1) is false. Then for an infinite subset of positive integers, Z_1, we have

$$n^{kp} \int_0^1 \sum_{i=1}^m w_i |f_i - s_{n,i}|^p < d^p \left(\int_0^1 \left(\sum w_i |f_i^{(k)}|^p \right)^{1/(kp+1)} \right)^{kp+1},$$

for some $0 < d < C_{k,r,p}$. We can find a $\bar{d}$ with $d < \bar{d} < C_{k,r}$ and closed subintervals $\{I_i\}$ so that $\sum_{i=1}^m w_i |f_i^{(k)}| \geqslant \delta > 0$, for some $\delta > 0$, and all $i = 1, ..., m$ with

$$d^p \left(\sum_{i=1}^m w_i \int_0^1 |f_i^{(k)}|^p \right)^{kp+1} \leqslant \bar{d}^p \left(\sum_{i=1}^m w_i \int_{[0,1]/A} |f_i^{(k)}|^p \right)^{kp+1}.$$

By arguments analogous to those given in case (ii) (see also [10, pp. 419–20]), we arrive at a contraction. That completes the proof of the theorem.

Acknowledgment

The author thanks the referee for his helpful suggestions.

References

1. M. Abramowitz and I. A. Stegun, (Eds.), "Handbook of Mathematical Functions," Dover, New York, 1965.
2. D. L. Barrow and P. W. Smith, Asymptotic properties of best $L_2[0, 1]$ approximation by splines with variable knots, *Quart. Appl. Math.* **36** (1978), 293–304.
3. D. L. Barrow and P. W. Smith, Efficient L_2 approximation by splines, *Number. Math.* **33** (1979), 101–114.
4. B. D. Bojanov, Uniqueness of the optimal nodes of quadrature formulae, *Math. Comp.* **36** (1981), 525–546.
5. H. G. Burchard and D. F. Hale, Piecewise polynomial approximation on optimal meshes, *J. Approx. Theory* **14** (1975), 128–147.
6. W. L. Chung, Automatic curve fitting using an adaptive local algorithm, *ACM Trans. Math. Software* **6** (1980), 45–57.
7. W. Dahmen and P. W. Smith, "A One Pass Curve Fitting Algorithm," Universitat Bonn, preprint No. 490 (1981).
8. M. Golomb, Approximation by periodic spline interpolants on uniform meshes, *J. Approx. Theory* **1** (1968), 26–65.
9. D. D. Pence, Asymptotic properties of approximation by splines with multiple knots, *in* "Approximation IV" (C. K. Chui, L. L. Schumaker, and J. D. Ward, Eds.), pp. 653–656, Academic Press, New York, 1983.
10. D. D. Pence and P. W. Smith, Asymptotic properties of best $L_p[0, 1]$ approximation by splines, *SIAM J. Math. Anal.* **13** (1982), 409–420.
11. I. J. Schoenberg, Monosplines and quadrature formulae, *in* "Theory and Applications of Spline Functions" (T. N. E. Greville, Ed.), pp. 157–207, Academic Press, New York, 1969.
12. L. L. Schumaker, "Spline Functions: Basic Theory," Wiley–Interscience, New York, 1981.
13. A.A. Zensykbaev, On the best quadrature formula on the class W^rL_p, *Soviet Math. Dokl.* **17** (1976), 377–380.